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On Tchebycheffian Spline Functions*

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1. INTRODUCTION

Because of a host of excellent constructive properties (such as a convenient, numerically stable basis; strong zero, sign change, and determinant properties; and optimal approximation power), certain finite dimensional linear spaces of piecewise polynomials (spline functions) have found an increasingly large number of numerical applications in recent years. These include, for example, methods for data fitting (such as interpolation, least squares, etc.) and numerical methods for operator equations (such as the Rayleigh–Ritz–Galerkin, collocation, least squares and other methods for eigenvalues, boundary-value problems, and control problems). For the bulk of these applications, polynomial spline functions are ideal.

On the other hand, there are some natural applications where a nonpolynomial piecewise structure may be preferable. For example, in certain data fitting problems it may be desirable to work with piecewise trigonometric or piecewise exponential functions. To give a more specific example, Reddien [24] has recently investigated a collocation method for numerical solution of a certain class of singular two-point boundary-value problems in which a space of functions which are piecewise linear combinations of nonpolynomial functions is used.

Although there is an extensive theory of generalized splines arising as solutions of best interpolation problems, there has been relatively little

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direct study of linear spaces of piecewise nonpolynomial functions. The first paper to discuss a general space of piecewise functions (see Remark 1 in Section 10 for some references dealing with specific nonpolynomial splines) is that of Greville [4]. Here a linear space of functions belonging piecewise to a fixed m dimensional linear space U_m and with global smoothness C^{m-2} was considered. With the help of a certain generalized Green's function, Greville was able to construct a basis for these splines. The next step was taken by Karlin and Ziegler [17], who defined a class of Tchebycheff splines, assuming that U_m is spanned by an Extended Complete Tchebycheff system (see Section 2). Further constructive properties of Tchebycheffian splines were obtained by Karlin [8], where local support bases (B-splines) and sign regularity properties of matrices formed from them were established. Some zero properties of Tchebycheffian splines were established in my dissertation [27], (cf. Karlin and Schumaker [15]).

The main purpose of this paper is to obtain basic structural results for an extended class of Tchebycheffian spline functions. In particular, we want to remove the hypothesis that the functions spanning U_m have to be $m - 1$ times differentiable. Thus, we shall work with a wide class of CT systems (which we shall call Canonical Complete Tchebycheff (CCT-) systems) instead of with the usual ECT systems. We believe this extension significantly enhances the applicability of Tchebycheffian splines, and in fact, one of the motivations for this paper was the desire to derive a local basis for numerical computation with the splines used by Reddien [24] for certain singular boundary-value problems.

In Section 2 we begin with the definition and basic properties of Canonical Complete Tchebycheff systems. The space of splines of interest in this paper is defined in Section 3. In later sections we discuss bases, B-splines, zero properties, a Green's function, and sign regularity properties of matrices formed from the spline bases. A central role in the development is played by the zero properties, which are new even for the case of ECT systems (cf. [29], where zero properties and applications for polynomial splines are discussed). This approach permits natural and direct proofs of the determinental and sign regularity results. The Tchebycheffian splines discussed by earlier authors are, of course, a special case, and the paper provides a new (and we believe simpler) development of their constructive properties. A specific example useful in [24] is studied in Section 9.

The question of approximation order using Tchebycheffian splines is not treated here. For direct theorems we refer to the papers of Jerome [5] and Jerome and Schumaker [7], and for inverse theorems to DeVore and Richards [3]. We conclude the paper with a section including remarks and some further references.

2. Canonical Complete Tchebycheff Systems

Suppose that u_1 is a bounded positive function on the interval $[a, b]$, and that $\sigma_2, ..., \sigma_m$ are bounded, right continuous, monotone increasing functions on $[a, b]$. Define

$$\begin{aligned} u_2(t) &= u_1(t) \int_a^t d\sigma_2(s_2), \\ &\vdots \\ u_m(t) &= u_1(t) \int_a^t \cdots \int_a^{s_{m-1}} d\sigma_m(s_m) \cdots d\sigma_2(s_2). \end{aligned} \tag{2.1}$$

We call any system of functions $\{u_i\}_1^m$ which can be written in this form a *Canonical Complete Tchebycheff* (CCT-) system. In this section we shall establish a number of properties of CCT-systems.

First, we should point out the connection of CCT-systems with the extensive hierarchy of Tchebycheff systems in the literature (cf. [8, 16]). If each of the σ's has the form

$$\sigma_i(t) = \int_a^t w_i(s)\, ds, \qquad i = 2, ..., m,$$

and if $w_i \in C^{m-i+1}[a, b]$, $i = 1, ..., m$ (where we set $w_1 = u_1$), then the system $\{u_i\}_1^m$ is the usual Extended Complete Tchebycheff (ECT-) system. The collection of CCT-systems is, of course, a much broader class. In one way it is even larger than the class of usual CT-systems defined in [8, 16] in that we are not even assuming the functions are continuous. On the other hand (contrary to an assertion of Rutman [25]), not every CT-system admits of a canonical representation of the form (2.1). (For a simple counterexample, see [35].) In any case, CCT-systems include many interesting examples; for one, see Section 9.

Two questions which are of considerable difficulty in the theory of Tchebycheff systems can be answered very easily for CCT-systems.

Lemma 2.1. *Suppose $\{u_i\}_1^m$ is a* CCT-*system on an interval $[a, b]$. Then each of these functions can be extended to form a* CCT-*system on any interval $[c, d]$ containing $[a, b]$.*

Proof. We need only extend u_1 to remain positive and bounded on $[c, d]$, and then extend each of the σ's to be bounded, right continuous, and monotone increasing. ▌

Lemma 2.2. *Suppose $\{u_i\}_1^m$ is a* CCT-*system on an interval $[a, b]$. Then there exists a function u_{m+1} such that the set $\{u_i\}_1^{m+1}$ is also a* CCT-*system on $[a, b]$.*

Proof. Choose any bounded, monotone increasing, right continuous function σ_{m+1}, and set

$$u_{m+1}(t) = u_1(t) \int_a^t \int_a^{s_2} \cdots \int_a^{s_m} d\sigma_{m+1}(s_{m+1})\, d\sigma_m(s_m) \cdots d\sigma_2(s_2). \quad \blacksquare$$

CCT-systems have the advantage that their structure allows us to develop properties along the same lines as for the ECT-systems. First, it is convenient to introduce certain reduced systems. If $\{u_i\}_1^m$ is a CCT-system, we define its jth reduced system by

$$\begin{aligned} v_{j,1}(t) &= 1, \\ v_{j,2}(t) &= \int_a^t d\sigma_{j+2}(s_{j+2}), \\ &\vdots \\ v_{j,m-j}(t) &= \int_a^t \cdots \int_a^{s_{m-1}} d\sigma_m(s_m) \cdots d\sigma_{j+2}(s_{j+2}). \end{aligned} \tag{2.2}$$

It will be convenient to introduce the notation

$$U_m^{(j)} = \operatorname{span} \{v_{j,i}\}_{i=1}^{m-j}. \tag{2.3}$$

Clearly, each of the reduced systems is also a CCT-system in its own right. We also observe that $u_i = v_{0,i}$, $i = 1, 2,\dots, m$. To see the connection between the u_i and the other reduced systems, we need to introduce certain "differential operators."

Suppose $\{u_i\}_1^m$ is a CCT-system on an interval $[a, b]$, and that $\{\sigma_i\}_2^m$ are the associated Stieltje's measures. Assuming that all of these functions have been extended to an interval to the right of b, then for all $\varphi \in U_m$ and all $a \leqslant t \leqslant b$, we may define

$$\begin{aligned} D_0\varphi(t) &= \varphi(t)/u_1(t), \\ D_j\varphi(t) &= \lim_{\delta \downarrow 0} \frac{\varphi(t+\delta) - \varphi(t)}{\sigma_{j+1}(t+\delta) - \sigma_{j+1}(t)}, \qquad j = 1, 2,\dots, m-1. \end{aligned} \tag{2.4}$$

Set

$$L_j = D_j \cdots D_1 D_0, \qquad j = 0, 1,\dots, m-1. \tag{2.5}$$

Now we may observe that

$$\begin{aligned} L_j u_i &= 0, & i &= 1,\dots, j, \\ &= v_{j,i-j}, & i &= j+1,\dots, m. \end{aligned} \tag{2.6}$$

It is also useful to note that

$$L_j u_i(t)\,|_{t=a} = \delta_{j,i-1}\,, \qquad j = 0, 1,..., i-1. \tag{2.7}$$

We turn now to results on determinants formed from a CCT-system. We need some notation. Given $u \in U_m$ and points

$$\{a \leqslant t_1 \leqq \cdots \leqq t_m \leqslant b\} = \{\tau_1\,,..., \tau_1\,,..., \tau_d\,,..., \tau_d\}, \tag{2.8}$$

where each τ_i is repeated exactly l_i times ($\sum_1^d l_i = m$), we define the matrix

$$M\begin{pmatrix} u_1\,,..., u_m \\ t_1\,,..., t_m \end{pmatrix} = \begin{bmatrix} u_1(\tau_1) & \cdots & u_m(\tau_1) \\ L_1u_1(\tau_1) & \cdots & L_1u_m(\tau_1) \\ \vdots & & \\ L_{l_1-1}u_1(\tau_1) & \cdots & L_{l_1-1}u_m(\tau_1) \\ \vdots & & \\ u_1(\tau_d) & \cdots & u_m(\tau_d) \\ \vdots & & \\ L_{l_d-1}u_1(\tau_d) & \cdots & L_{l_d-1}u_m(\tau_d) \end{bmatrix} \tag{2.9}$$

(cf. [16, p. 5] for a similar definition for the ECT-system case). We denote the determinant of this matrix with the letter D.

In view of (2.6), it is clear that for any $t \in [a, b]$,

$$D\begin{pmatrix} u_1\,,..., u_m \\ t,..., t \end{pmatrix} = \prod_{j=0}^{m-1} v_{j,1}(t) \equiv u_1(t) > 0. \tag{2.10}$$

(This is the analog of the fact that the Wronskian of an ECT-system is always positive.) We can now show the positivity of all determinants formed with t's as in (2.8).

THEOREM 2.3. *Suppose that* $\{u_i\}_1^m$ *is a* CCT-*system and that* $\{t_i\}_1^m$ *are as in* (2.8). *Then*

$$D\begin{pmatrix} u_1\,,..., u_m \\ t_1\,,..., t_m \end{pmatrix} > 0, \tag{2.11}$$

where D is the determinant of the matrix defined in (2.9).

Proof. We proceed by induction on m. For $m = 1$, the statement is precisely the statement (2.10). Now suppose that the result has been proved for CCT-systems of $m - 1$ functions. We shall proceed along the lines of proof of the analogous result for ECT-systems (see [16, p. 377]). The key here is the fact (cf. (2.1), (2.2), (2.6)) that

$$u_j/u_1(t_2) - u_j/u_1(t_1) = \int_{t_1}^{t_2} L_1u_j(s)\, d\sigma_2(s), \qquad 2 \leqslant j \leqslant m. \tag{2.12}$$

Now, starting with the determinant (2.11), we shall reduce it to an integral of a similar $m - 1$ square determinant over a region on which it is strictly positive. First, we observe by (2.6) that the only nonzero entries in the first column correspond to the d rows starting with $u_1(\tau_1)$, with $u_1(\tau_2)$, etc., up to $u_1(\tau_d)$. Suppose we factor $u_1(\tau_i)$ out of the ith of these rows for $i = 1, 2,..., d$. Now subtract the $(d - 1)$st from the dth, the $(d - 2)$nd from the $(d - 1)$st, etc., and the 2nd from the first. We now have a 1 in the upper left-hand corner, and the rest of column 1 is 0. Expanding, we have an $m - 1$ square determinant. In the rows corresponding to $\tau_2 ,..., \tau_d$, we have expressions as in (2.12). Using (2.12), we obtain that

$$D\begin{pmatrix} u_1 ,..., u_m \\ t_1 ,..., t_m \end{pmatrix} = \int_{\tau_1}^{\tau_2} \int_{\tau_2}^{\tau_3} \cdots \int_{\tau_{d-1}}^{\tau_d} \varphi(s_1 ,..., s_{d-1})\, d\sigma_2(s_1) \cdots d\sigma_2(s_{d-1}),$$

where

$$\varphi(s_1 ,..., s_{d-1}) = D\begin{pmatrix} L_1u_2 ,..., L_1u_m \\ \tau_1 ,..., \tau_1 , s_1 , \tau_2 ,..., \tau_2 , s_2 ,..., s_{d-1} , \tau_d ,..., \tau_d \end{pmatrix}$$

with τ_i appearing exactly $l_i - 1$ times, $i = 1, 2,..., d$. Since σ_i are monotone increasing, there is mass in each of the stated intervals, and the integral is positive, as its integrand is nonnegative on the closed interval and positive in the interior. ▮

We close this section with a statement about the number of zeros any nontrivial $u \in U_m$ may possess. Because of the special structure of CCT-systems, we may define a kind of multiple zero for them. We say u has a z-tuple zero at the point t in $[a, b]$ provided that

$$u(t) = L_1u(t) = \cdots = L_{z-1}u(t) = 0 \neq L_zu(t) \tag{2.13}$$

if $1 \leqslant z \leqslant m - 1$. We say u has an m-tuple zero at t if all of the expressions $u(t),..., L_{m-1}u(t)$ vanish.

THEOREM 2.4. *Suppose U_m is spanned by a* CCT-*system, and that Z counts the number of zeros of an element $u \in U_m$ according to the rule* (2.13). *Then if u is not identically zero,*

$$Z(u) \leqslant m - 1. \tag{2.14}$$

Proof. Suppose $u = \sum_1^m \alpha_i u_i$ has m zeros, say at points $\{t_i\}_1^m$ as in (2.8). Then the vector $\alpha = (\alpha_1 ,..., \alpha_m)^T$ must satisfy the linear system

$$M\begin{pmatrix} u_1 ,..., u_m \\ t_1 ,..., t_m \end{pmatrix} \alpha = 0,$$

which is nonsingular by Theorem 2.3. Thus $\alpha = 0$. ▮

3. Tchebycheffian Splines

In this section we define the space of splines of interest in this paper, and determine its dimension. Let $a = x_0 < x_1 < \cdots < x_{k+1} = b$, and set $\Delta = \{x_i\}_1^k$. The set Δ partitions the interval $[a, b]$ into $k + 1$ subintervals $I_i = [x_i, x_{i+1})$, $i = 0, 1,..., k - 1$ and $I_k = [x_k, x_{k+1}]$. Let $1 \leqslant m_i \leqslant m$ be integers, and define $\mathscr{M} = (m_1, ..., m_k)$. Finally, let U_m be spanned by a Canonical CT-system, and let $\{L_j\}_0^{m-1}$ be the operators defined in (2.5).

We call the space

$$\begin{aligned}\mathscr{S}(U_m; \mathscr{M}; \Delta) = \{s : s_i = s\,|_{I_i} \in U_m,\ i = 0, 1,..., k \text{ and}& \\ L_j s_{i-1}(x_i) = L_j s_i(x_i),\ j = 0, 1,..., m - 1 - m_i \text{ for } i = 1,..., k\}&\end{aligned} \tag{3.1}$$

the space of *Tchebycheffian spline functions with knots at* $x_1, ..., x_k$ *of multiplicities* $m_1, ..., m_k$. It is clearly a linear space.

THEOREM 3.1. *Let* $K = \sum_1^k m_i$. *Then* $\mathscr{S}(U_m; \mathscr{M}; \Delta)$ *is of dimension* $m + K$.

Proof. Suppose $s_i(x) = \sum_{j=1}^m c_{ij} u_j(x)$ for $x \in I_i$, $i = 0, 1,..., k$. Then the continuity conditions relating successive pieces of s can be written in the form

$$Ac = \begin{bmatrix} A_1 & -A_1 & & & & \\ & A_2 & -A_2 & & & \\ & & \cdots & & & \\ & & & & A_k & -A_k \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_k \end{bmatrix} = 0,$$

where $c_i = (c_{i1}, ..., c_{im})^T$ and

$$A_i = \begin{bmatrix} L_0 u_1(x_i) & L_0 u_2(x_i) & & \cdots & & L_0 u_m(x_i) \\ 0 & L_1 u_2(x_i) & & \cdots & & L_1 u_m(x_i) \\ \vdots & & & & & \\ 0 & 0 & \cdots & L_{m-m_i-1} u_{m-m_i}(x_i) & \cdots & L_{m-m_i-1} u_m(x_i) \end{bmatrix}.$$

It is clear that A_i is of rank $m - m_i$. Hence, the matrix A has rank equal to the number of its rows, viz., $\sum_1^k (m - m_i)$. As A is a transformation of Euclidean $m(k + 1)$ space into Euclidean $\sum_1^k (m - m_i)$ space, the dimension of the null space of A is equal to $m(k + 1) - \sum_1^k (m - m_i) = m + \sum_1^k m_i = m + K$. This is the dimension of $\mathscr{S}$. ∎

The reader who is not too familiar with Tchebycheffian splines may want to keep in mind the case where $U_m = \mathscr{P}_m$ = space of polynomials of order m.

In this case an ECT system basis in the canonical form (2.1) is given by the functions $u_i(x) = x^{i-1}/(i-1)!$ with $w_i(x) = 1$, $i = 1, 2,..., m$. In this case the operator $L_j = D^j$, the usual jth derivative operator. (For a treatment of polynomial splines which more or less parallels the development here, see [29].)

4. A One-Sided Basis

Our aim in this section is to construct a basis for the space $\mathscr{S}(U_m; \mathscr{M}; \Delta)$. First, we observe that $U_m \in \mathscr{S}(U_m; \mathscr{M}; \Delta)$. Thus, as part of a basis we may take the m functions $u_1,..., u_m$. To complete a basis, in view of Theorem 3.1 we would now like to construct m_i splines associated with each knot x_i, $i = 1, 2,..., k$. To this end, we now introduce certain functions whose form is similar to the u_j, but with the point a replaced by an arbitrary point y in (a, b).

We define

$$\begin{aligned} g_1(t, y) &= 0 \qquad && t < y, \\ &= u_1(t), && t \geqslant y, \end{aligned} \tag{4.1}$$

and for $j = 2,..., m$

$$\begin{aligned} g_j(t, y) &= 0, && t < y, \\ &= u_1(t) \int_y^t \int_y^{s_2} \cdots \int_y^{s_{j-1}} d\sigma_j(s_j) \cdots d\sigma_2(s_2), \qquad && t \geqslant y. \end{aligned} \tag{4.2}$$

Clearly,

$$L_i g_j(t, y) \mid_{t=y} = \delta_{i,j-1}, \qquad i = 0, 1,..., j-1. \tag{4.3}$$

Moreover, for $t \geqq y$ the following lemma shows that $g_j(t, y) \in U_m$. In fact, we show that

$$g_j(t, y) \in U_j = \operatorname{span} \{u_i\}_1^j \qquad \text{for } t \geqslant y \tag{4.4}$$

and in fact, for $t \geqslant y$,

$$g_j(t, y) = u_j(t) + \cdots. \tag{4.5}$$

Lemma 4.1. *For $j = 2,..., m$ and all $t \geqslant y$,*

$$g_j(t, y) = \sum_{i=1}^{j} u_i(t)\, v^*_{m-j,j-i+1}(y)(-1)^{j-i}, \tag{4.6}$$

where

$$\begin{aligned} v_{j,1}^*(y) &= 1, \\ v_{j,2}^*(y) &= \int_a^y d\sigma_{m-j}(t_{m-j}), \\ &\vdots \\ v_{j,m-j}^*(y) &= \int_a^y \int_a^{t_{m-j-1}} \cdots \int_a^{t_3} d\sigma_2(t_2) \cdots d\sigma_{m-j}(t_{m-j}), \end{aligned} \tag{4.7}$$

all $j = 0, 1,..., m-1$.

Proof. We prove somewhat more; namely, for all $r = 0, 1,..., j-2$,

$$\int_y^t \int_y^{t_{r+2}} \cdots \int_y^{t_{j-1}} d\sigma_j(t_j) \cdots d\sigma_{r+2}(t_{r+2}) = \sum_{i=r+1}^{j} (-1)^{j-i} v_{r,i-r}(t)\, v_{m-j,j-i+1}^*(y). \tag{4.8}$$

We accomplish this by induction on r. For $r = j-2$,

$$\begin{aligned} \int_y^t d\sigma_j(t_j) &= \int_a^t d\sigma_j(t_j) - \int_a^y d\sigma_j(t_j) = v_{j-2,2}(t) - v_{m-j,2}^*(y) \\ &= v_{j-2,2}(t)\, v_{m-j,1}^*(y) - v_{j-2,1}(t)\, v_{m-j,2}^*(y), \end{aligned}$$

which is (4.8) in this case. Now we assume that (4.8) holds for $r+1,..., j-2$, and prove it for r. We have

$$\int_y^t \int_y^{t_{r+2}} \cdots \int_y^{t_{j-1}} d\sigma_j \cdots d\sigma_{r+2} = \int_a^t \phi(t_{r+2})\, d\sigma_{r+2} - \int_a^y \phi(t_{r+2})\, d\sigma_{r+2}, \tag{4.9}$$

where (using the induction hypothesis)

$$\begin{aligned} \phi(t_{r+2}) &= \int_y^{t_{r+2}} \int_y^{t_{r+3}} \cdots \int_y^{t_{j-1}} d\sigma_j \cdots d\sigma_{r+3} \\ &= \sum_{i=r+2}^{j} (-1)^{j-i} v_{r+1,i-r-1}(t_{r+2})\, v_{m-j,j-i+1}^*(y). \end{aligned}$$

Substituting this in the first term of (4.9), we see that it reduces to

$$\sum_{i=r+2}^{j} (-1)^{j-i} v_{r,i-r}(t)\, v_{m-j,j-i+1}^*(y).$$

Now a simple induction argument shows that

$$\int_a^y \int_y^{t_{r+2}} \cdots \int_y^{t_{j-1}} d\sigma_j \cdots d\sigma_{r+2} = (-1)^{j-r} \int_a^y \int_a^{t_j} \cdots \int_a^{t_{r+3}} d\sigma_{r+2} \cdots d\sigma_j .$$

Thus, since $v_{r,1}(t) = 1$, the second term in (4.9) is equal to

$$(-1)^{j-r-1} v_{r,1}(t)\, v^*_{m-j,j-r}(y),$$

and (4.8) is proved for all r. ▮

The meaning of the expansion (4.6) may be somewhat clearer if we observe that in the polynomial case it is just the binomial expansion:

$$g_j(t, y) = \frac{(t-y)_+^{j-1}}{(m-1)!} = \sum_{i=1}^{j} \frac{t^{i-1} y^{j-i} (-1)^{j-i}}{(i-1)!\,(j-i)!}, \qquad t \geqslant y.$$

THEOREM 4.2. *A basis for $\mathscr{S}(U_m\,; \mathscr{M}; \Delta)$ is given by*

$$\{\rho_{ij}(t) = g_{m-j+1}(t, x_i)\}_{j=1,i=0}^{m_i,k}, \tag{4.10}$$

where for convenience we have set $m_0 = m$.

Proof. First, observe that $g_{m-j+1}(t, x_0) = u_{m-j+1}(t)$, $j = 1, 2,..., m_0 = m$. As observed before, these functions belong to $\mathscr{S}$. Moreover, in view of properties (4.3) and (4.4), it also follows that the functions $g_m(t, x_i),...,$ $g_{m-m_i+1}(t, x_i)$ belong to $\mathscr{S}$, $i = 1, 2,..., k$. Now by Theorem 3.1 we have the right number of functions for a basis. It remains only to prove that they are linearly independent.

Suppose that

$$\sum_{i=0}^{k} \sum_{j=1}^{m_i} c_{ij} \rho_{ij}(t) = 0.$$

Then for all $t \in I_0 = [x_0\,, x_1)$ this reduces to $c_{01}\rho_{01}(t) + \cdots + c_{0m}\rho_{0m}(t) = 0$. But $\{\rho_{0j}\}_1^m$ span U_m, so we conclude $c_{01} = \cdots = c_{0m} = 0$. Next we consider $t \in I_1 = [x_1\,, x_2)$. Now we have $c_{11}\rho_{11}(t) + \cdots + c_{1m_1}\rho_{1m_1}(t) = 0$. But in view of (4.5), these functions are also linearly independent on I_1, so these c's are 0. The process may be continued to show that all c's are 0, and the linear independence is established. ▮

In closing this section we may observe that all of the splines used in Theorem 4.2 come from one fixed basic spline by the application of appropriate "differentiation" operators. In particular, if we define $L_j^* = D_j^* \cdots D_1^*$, where

$$D_j^* \varphi(t) = \lim_{\delta \downarrow 0} \frac{\varphi(t) - \varphi(t-\delta)}{\sigma_{m-j+1}(t) - \sigma_{m-j+1}(t-\delta)}, \qquad j = 1, 2,..., m-1 \tag{4.11}$$

(we emphasize that these are *not* the adjoints of the L_j defined earlier), then

$$\begin{aligned} L_j{}^* v_{0,i}^* &= 0, \qquad & i &= 1, 2,\ldots, j, \\ &= v_{j,i-j}^*, \qquad & i &= j+1,\ldots, m. \end{aligned} \tag{4.12}$$

Then, if we apply $L_j{}^*$ to the y variable in the expansion

$$\begin{aligned} g_m(t, y) &= \sum_{i=1}^{m} u_i(t)\, v_{0,m-i+1}^*(y)(-1)^{m-i}, \qquad & t &\geqslant y, \\ &= 0, & t &< y, \end{aligned} \tag{4.13}$$

we obtain (using Lemma 4.1 again),

$$g_{m-j}(t, y) = (-1)^j L_j{}^* g_m(t, y), \qquad j = 1, 2,\ldots, m-1. \tag{4.14}$$

Thus all of the splines can be obtained from the "Green's function" $g_m(t, y)$. We discuss some further properties of this Green's function in Section 8.

5. Zero Properties

Up to this point, we have made no assumption that the underlying CCT-system involved in the definition of splines should be continuous. For the remainder of the paper, however, we will have to work with splines which are continuous (along with their "derivatives" $L_j s$). To ensure this, we assume henceforth that u_1 and all of the σ's in the canonical expansion (2.1) are continuous.

In this section we shall show that a nontrivial spline $s \in \mathscr{S}(U_m\,; \mathscr{M}; \Delta)$ can have at most $m + K - 1$ zeros, counting multiplicities in a very strong way. Before we can state this result, we need to agree on how to count multiplicities.

Let $s \in \mathscr{S}(U_m\,; \mathscr{M}; \Delta)$. Then since between any two knots, s is an element of U_m, we know from Theorem 2.4 that either it vanishes identically throughout this interval, or it can be zero only at a finite number of isolated points in the interval. The multiplicities of isolated zeros of s at points $t \notin \Delta$ will be counted exactly as in (2.13).

When s vanishes identically on an interval between two knots, then we count the entire interval as either m or $m + 1$ according to the following rules:

> If $s(t) = 0$ for $[a, x_j)$ but $s(t) \neq 0$ for $x_j < t < x_j + \epsilon$ for some $\epsilon > 0$, then we count $[a, x_j]$ as an interval zero of s of multiplicity $z = m$. A similar count is used if s vanishes on an interval ending at b. (5.1)

If $s(t) = 0$ for all $[x_i, x_j]$ but $s(t) \neq 0$ for $x_i - \epsilon < t < x_i$ and $x_j < t < x_j + \epsilon$, some $\epsilon > 0$, then we count the multiplicity of $[x_i, x_j]$ as (5.2)

$$\begin{aligned} z &= m + 1, \quad \text{if } m \text{ is even and } s(x_i - \epsilon/2)\, s(x_j + \epsilon/2) < 0, \\ &= m + 1, \quad \text{if } m \text{ is odd and } s(x_i - \epsilon/2)\, s(x_j + \epsilon/2) > 0, \\ &= m, \quad \text{otherwise.} \end{aligned}$$

It remains to consider the case where s is zero at a knot, but not in an interval containing the knot, or where s jumps through zero at a knot. If $t \in \Delta$ and s does not vanish in any interval containing t, then we define the multiplicity of the zero t by

Suppose $t = x_i$, and s_{i-1} and s_i are the elements of U_m representing s on I_{i-1} and I_i (cf. (3.1)). Let $\alpha = \max(l, r)$ such that $s_{i-1}(t) = L_1 s_{i-1}(t) = \cdots = L_{l-1} s_{i-1}(t) = 0 \neq L_l s_{i-1}(t)$ and $s_i(t) = L_1 s_i(t) = \cdots = L_{r-1} s_i(t) = 0 \neq L_r s_i(t)$. Then we count the multiplicity of t as (5.3)

$$\begin{aligned} z &= \alpha + 1, \quad \text{if } \alpha \text{ is even and } s \text{ changes sign at } t, \\ &= \alpha + 1, \quad \text{if } \alpha \text{ is odd and } s \text{ does not change sign at } t, \\ &= \alpha, \quad \text{otherwise.} \end{aligned}$$

This rule counts a jump through 0 at a knot as a zero of multiplicity 1. The rules (5.2) and (5.3) have been designed so that s has a sign change at a zero of odd multiplicity, and no sign change at a zero of even multiplicity. We also note that the rule (5.3) is actually equivalent to the usual one (2.13) if we apply it to $t \notin \Delta$.

THEOREM 5.1. *Suppose that $\sigma_2, \ldots, \sigma_m$ are continuous (so that U_m and the reduced systems are continuous functions). Then*

$$Z(s) \leqq m + K - 1, \qquad \textit{all} \quad s \in \mathscr{S}(U_m; \mathscr{M}; \Delta), \qquad s \not\equiv 0, \tag{5.4}$$

where Z counts the number of zeros of s in $[a, b]$, with multiplicities as in (5.1)–(5.3).

Proof. For $m = 1$, Z simply counts the number of times that the piecewise function s jumps through 0. Since $u_1(t) > 0$ in $[a, b]$, such jumps can only occur at knots. It follows that $Z(s) \leqq k = K$ in this case.

To prove the theorem in general, we proceed by induction on the order of U_m. Suppose the theorem is true for order $m - 1$, and in particular for splines associated with the space $U_m^{(1)}$ spanned by the first reduced system (see (2.3)). We shall now prove the result for U_m.

First, suppose that $\mathscr{M}$ is such that there is no $m_i = m$. Then s is continuous. Consider the function $\tilde{s} = L_1 s$. It is clear that $\tilde{s}$ is actually a spline in the space $\mathscr{S}(U_m^{(1)}; \mathscr{M}'; \Delta)$, where $\mathscr{M}' = (m_1', ..., m_k')$ with $m_i' = \min(m_i, m - 1)$. Now we shall show that $Z(s) \geqq m + K$ leads to a contradiction by showing that in this case $\tilde{Z}(\tilde{s}) \geqq m + K - 1$, which is impossible by the induction hypothesis. (Note: $\tilde{Z}$ counts multiplicities with respect to the CCT system spanning $U_m^{(1)}$, which uses different operators.)

By the definition of multiple zero (2.13), it follows that if s has a zero of multiplicity $z > 1$ at a point $t \notin \Delta$, then $\tilde{s}$ has a zero of multiplicity $z - 1$ at the same point. The same is true for definitions (5.1)–(5.3). For example, for (5.3) we have the following table:

α	s changes sign	$Z(s)$	$\alpha - 1$	$\tilde{s}$ changes sign	$\tilde{Z}(\tilde{s})$
even	yes	$\alpha + 1$	odd	no	α
even	no	α	odd	yes	$\alpha - 1$
odd	yes	α	even	no	$\alpha - 1$
odd	no	$\alpha + 1$	even	yes	α

In addition to the zeros of $\tilde{s}$ coming from multiple zeros of s, we also observe that by the continuity of s, between any two zeros of s, $\tilde{s}$ must have a change of sign. Thus, assuming that there are a total of d points and intervals where s vanishes with multiplicities $z_1, ..., z_d$ and $Z(s) = \sum_1^d z_i = m + K$, we find that

$$\tilde{Z}(\tilde{s}) \geqslant \sum_1^d (z_i - 1) + d - 1 = m + K - 1.$$

This is our desired contradiction, and we conclude that no s with $m + K$ zeros can exist.

It remains to consider the case where some of the knots are m-tuple. Suppose for the moment that there is only one such knot, say x_l with $m_l = m$. Define s_R to be the restriction of s to $[x_l, b]$, and s_l to be the restriction of s to $[a, x_l)$. We define $s_L(x_l) = \lim_{t \uparrow x_l} s_L(t)$. Then by what we have already proved, we have

$$Z_{[a,x_l]}(s_L) \leqslant m + \sum_1^{l-1} m_i - 1 \qquad \text{and} \qquad Z_{[x_l,b]}(s_R) \leqslant m + \sum_{l+1}^{k} m_i - 1.$$

Moreover, we note that if x_l is a zero of s of multiplicity z_l, then by the definition of multiplicity, x_l is also a zero of multiplicity $z_l - 1$ of either s_L or s_R. Thus, we conclude that

$$Z(s) \leqslant Z_{[a,x_l]}(s_L) + Z_{[x_l,b]}(s_R) + 1 \leqslant m + \sum_1^k m_i - 1.$$

If there are several knots of multiplicity m, we simply divide $[a, b]$ into the corresponding number of pieces and argue in the same way. ∎

If U_m is a CCT system on a larger interval $[c, d] \supset [a, b]$, then each element $s \in \mathscr{S}(U_m; \mathscr{M}; \Delta)$ has a natural extension to $[c, d]$ defined by taking s_0 throughout $[c, x_1)$ and s_k throughout $[x_k, d]$. Since we did not use any properties relating to $[a, b]$, it is clear that Theorem 5.1 remains valid if we count zeros as above, but throughout the entire interval $[c, d]$.

6. Local Bases and B-Splines

For numerical applications the one-sided basis constructed in Section 4 is generally not well conditioned. It would be much preferable to have a local support basis. In this section we shall construct one by constructing analogs of the B-splines.

In view of Theorem 4.2, it would be natural to try to construct a local support basis for $\mathscr{S}$ by taking linear combinations of the one-sided splines in (4.10). The following lemma gives necessary and sufficient conditions for a linear combination of such one-sided splines to have local support.

Lemma 6.1. *Let* $a \leqslant z_l < z_{l+1} < \cdots < z_r < b$ *and* $1 \leqslant \mu_i \leqslant m$, $i = l, l+1, \ldots, r$, *and suppose*

$$B(t) = \sum_{i=l}^{r} \sum_{j=1}^{\mu_i} \alpha_{ij}\, g_{m-j+1}(t, z_i).$$

Then $B(t) \equiv 0$ *for* $t \geqslant z_r$ *if and only if*

$$\sum_{i=l}^{r} \sum_{j=1}^{\mu_i} \alpha_{ij} L_{j-1}^{*} v_{0,m-\nu+1}^{*}(z_i)(-1)^{j-1} = 0, \qquad \nu = 1, 2, \ldots, m. \tag{6.1}$$

In particular, if $B(t) \equiv 0$ *for* $t \geqslant z_r$, *then* B *can be nontrivial if and only if* $\sum_l^r \mu_i > m$.

Proof. First, we observe that by (4.13) and (4.6), for $t \geqslant z_r$,

$$g_{m-j+1}(t, z_i) = (-1)^{j-1} \sum_{\nu=1}^{m-j+1} u_\nu(t)\, L_{j-1}^{*} v_{0,m-\nu+1}^{*}(z_i)(-1)^{m-\nu}.$$

By (4.12) we may write the sum to m as all the extra terms are 0. Now, interchanging the order of summation and using the linear independence of the $u_1, \ldots, u_m$ we conclude (6.1) must hold. The converse is clear.

Now if $\sum_l^r \mu_i \leqslant m$, then the first $\sum_l^r \mu_i$ equations of (6.1) are a homogeneous system for the $\sum_l^r \mu_i$ coefficients. It is nonsingular by Theorem 2.3 (applied to the CCT system $\{v_{0,j}^*\}_1^m$). This implies B is trivial.

An alternate proof of the second assertion here can be based on the zero theorem of Section 5. If B vanishes both to the left and right of (z_l, z_r), then it has $2m$ zeros. Hence, it must have at least $m + 1$ knots. ∎

The simplest case where we can hope to construct a local support spline is the case where $\sum_l^r \mu_i = m + 1$ in Lemma 6.1. In this case, by Cramer's rule, the solution of (6.1) must be such that up to a constant multiple,

$$B(t) = D^* \begin{pmatrix} u_1^*, \ldots, u_m^*, g_m(t, \cdot) \\ x_l, \ldots, x_l, \ldots, x_r, \ldots, x_r \end{pmatrix},$$

where we have written $u_i^* = v_{0,i}^*$, $i = 1, 2, \ldots, m$, for convenience, and where the $*$ on the determinant is to remind us that multiplicities are to be treated as in (2.2), but using the operators L_j^*, $j = 1, 2, \ldots, m - 1$.

The determinant defining $B(t)$ may be regarded as a generalized divided difference (cf. [21] and references therein). Indeed, if $\{u_i^*\}_1^m$ is extended to a CCT system $\{u_i^*\}_1^{m+1}$ by the addition of one function, then given any $t_1 \leqslant \cdots \leqslant t_{m+1}$ and a function f for which the required "derivatives" exist, we define the *divided difference of f over $t_1, \ldots, t_{m+1}$ with respect to $\{u_i^*\}_1^{m+1}$* by

$$[t_1, \ldots, t_{m+1}]f = \frac{D^* \begin{pmatrix} u_1^*, \ldots, u_m^*, f \\ t_1, \ldots, t_m, t_{m+1} \end{pmatrix}}{D^* \begin{pmatrix} u_1^*, \ldots, u_{m+1}^* \\ t_1, \ldots, t_{m+1} \end{pmatrix}}. \tag{6.2}$$

It is not hard to show that this divided difference has the properties of the usual one. For example, for all $u \in U_m^* = \operatorname{span}\{u_i^*\}$ the divided difference is 0. Indeed, this definition coincides with the usual one if we take $u_i^*(t) = t^{i-1}/(i-1)!$.

Now that we have succeeded in constructing a local support spline, we can construct a local support basis for $\mathscr{S}(U_m; \mathscr{M}; \Delta)$.

THEOREM 6.2. *Suppose $\{u_i^*\}_1^m$ is a CCT system on an interval $[c, d]$ containing $[a, b]$. Let $y_{m+1} \leqslant \cdots \leqslant y_{m+K}$ be an enumeration of the sequence $x_1, \ldots, x_1, \ldots, x_k, \ldots, x_k$, where each x_i is repeated exactly m_i times, $i = 1, 2, \ldots, k$. Let $c < y_1 \leqslant \cdots \leqslant y_m \leqslant a$ and $b \leqslant y_{m+K+1} \leqslant \cdots < y_{2m+K} < d$ be arbitrary. Define*

$$B_i(t) = [y_i, \ldots, y_{i+m}]\, g_m(t, \cdot), \qquad i = 1, 2, \ldots, m + K. \tag{6.3}$$

Then $\{B_i\}_1^{m+K}$ *is a basis for* $\mathscr{S}(U_m ; \mathscr{M}; \Delta)$. *Moreover,*

$$B_i(t) > 0 \qquad on \quad (y_i, y_{i+m}), \tag{6.4}$$

and

$$B_i(t) = 0 \qquad for \quad t < y_i, \qquad y_{i+m} \leqslant t, \tag{6.5}$$

$i = 1, 2,..., m + K$.

Proof. By the definition of divided difference and the observation (4.13), it follows that each of the B_i is a linear combination of the $\{\rho_{ij}\}_{j=1,i=0}^{m_i;k}$ of Theorem 4.2. Hence, each $B_i \in \mathscr{S}(U_m ; \mathscr{M}; \Delta)$. As there are the right number of B's, we will have proved they are a basis if we show they are linearly independent. We defer this until the following lemma, where a strong form of linear independence is established.

By the one-sided property of the ρ_{ij}'s, it follows automatically that $B_i(t) = 0$ for $a \leqslant t < y_i$. For $y_{i+m} \leqslant t \leqslant b$, we have the $(m + 1)$st divided difference of an element in $U_m{}^*$ (cf. (4.13)), which as remarked above is 0. Property (6.4) follows from Lemma 6.4 below which gives an even stronger property of the B-splines. ▮

Note that the assumption that $\{u_i{}^* = v_{0,i}^*\}_1^m$ is a CCT system on an interval $[c, d]$ larger than $[a, b]$ is no restriction as CCT systems can always be extended as in Lemma 2.1.

LEMMA 6.3. *Let* $m \leqslant l < r \leqslant m + K + 1$, *and suppose* $y_l < y_{l+1}$ *and* $y_{r-1} < y_r$. *Then* $\{B_i\}_{l-m+1}^{r-1}$ *are linearly independent on* $[y_l, y_r]$. (*These are precisely the B-splines with support on this interval.*)

Proof. We may choose $y_l < t_{l-m+1} < \cdots < t_{r-1} < y_r$ so that $t_i \in (y_i, y_{i+m})$, $i = l - m + 1,..., r - 1$. Now by Corollary 7.3,

$$D\begin{pmatrix} B_{l-m+1}, ..., B_{r-1} \\ t_{l-m+1}, ..., t_{r-1} \end{pmatrix} > 0,$$

which, of course, implies the linear independence. ▮

Lemma 6.3 asserts that the B-splines $\{B_i\}_{l-m+1}^{l}$ form a basis for U_m on the interval $[y_l, y_{l+1}]$. We close this section with a sharper result on the behavior of B_i in the interval (y_i, y_{i+m}).

THEOREM 6.4. *For all* $i = 1, 2,..., m + K$ *and all* $j = 0, 1,..., m - 1$

$$Z_{(y_i, y_{i+m})}(L_j B_i) \leqslant j.$$

Proof. We note that $L_j B_i \in \mathcal{S}(U_m^{(j)}; \mathcal{M}^{(j)}; \Delta)$, with $\mathcal{M}^{(j)} = (m_1^{(j)},..., m_k^{(j)})$ and $m_i^{(j)} = \min(m_i, m - j)$, $i = 1, 2,..., k$ (cf. the proof of Theorem 5.1). Then Theorem 5.1 guarantees that $Z(L_j B_i) \leqq 2m - j$. Since $L_j B_i$ has $m - j$ zeros on each end (it is order $m - j$), it can have at most j in (y_i, y_{i+m}). ∎

7. The B-Spline Collocation Matrix

Suppose $a \leqslant t_1 \leqslant \cdots \leqslant t_{m+K} \leqslant b$ are real numbers with at most m of them equal to any one value. In this section we shall consider the matrix

$$M \begin{pmatrix} B_1, ..., B_{m+K} \\ t_1, ..., t_{m+K} \end{pmatrix}$$

defined as in (2.9) and its determinant D. We assume throughout this section that the σ's defining the CCT-system are continuous so that each function in U_m is continuous (cf. Theorem 5.1).

THEOREM 7.1. *Let $m > 1$. Suppose at most m of the t's and y's (the knots of the B's) take on any one value. Then*

$$D \begin{pmatrix} B_1, ..., B_{m+K} \\ t_1, ..., t_{m+K} \end{pmatrix} \neq 0$$

if and only if

$$y_i < t_i < y_{i+m}, \qquad i = 1, 2,..., m + K. \tag{7.1}$$

Proof. It is easily seen using Laplace's expansion that if (7.1) fails to hold, then $D = 0$ (cf. [8, 17]). Now suppose that (7.1) holds, but that the determinant is nevertheless 0. Then there exists a nontrivial linear combination of the B's, say $s = \sum_1^{m+K} c_i B_i$, which vanishes at all of the t's, along with "derivatives" $L_1 s$,... in case of multiplicities. Let c_l be the first nonzero coefficient, and suppose $l \leqslant r \leqslant m + K$ is the smallest index so that s is 0 on an interval with left endpoint y_{r+m}. Then $\tilde{s} = \sum_l^r c_i B_i$ has an m-tuple zero on $[c, y_l)$ and an m-tuple zero on $[y_{r+m}, d]$. In addition, it vanishes at the points $t_l ,..., t_r$ which lie in (y_l, y_{r+m}) by (7.1). As s does not vanish on an interval in (y_l, y_{r+m}), we see that it has a total of $2m + r - l + 1$ zeros. But it only has $m + r - l + 1$ knots, contradicting Theorem 5.1. The determinant cannot be 0. ∎

The conditions (7.1) require that each t_i lie in the interior of the support of the corresponding B-spline B_i, $i = 1, 2,..., m + K$.

COROLLARY 7.2. *Under the conditions* (7.1) *the determinant in Theorem* 7.1 *is positive.*

Proof. Suppose first that $y_m < \cdots < y_{m+K}$. Now for any choice of $t_1 < \cdots < t_{m+K}$ such that each t_i lies in the support of B_i but not in the support of any other B-spline, $i = 1, 2, ..., m + K$, it is clear that the matrix of interest is diagonal with positive entries on the diagonal, and hence the determinant is positive. (There do exist such t's.) Now since all of the B-splines are continuous functions of t, the entire matrix is a continuous function of the vector $(t_1, ..., t_{m+K})$ as long as we keep the t's distinct. We conclude that as this vector runs over all distinct t's satisfying (7.1), the determinant is always positive (as by Theorem 7.1 it never vanishes). Now, if we let the t's coalesce, the sign of the determinant does not change. We conclude that the determinant is positive for all $t_1 \leqslant \cdots \leqslant t_{m+K}$ satisfying (7.1).

To complete the proof, suppose that $\{t_i\}_1^{m+K}$ satisfy (7.1), and that the $\{y_i\}_1^{2m+K}$ include possible multiplicities. Let $y_1^{(\nu)} \leqslant \cdots \leqslant y_{2m+K}^{(\nu)}$ be a sequence of distinct $y^{(\nu)}$'s converging (say from above) to the $y_1, ..., y_{2m+K}$. For sufficiently large ν, (7.1) will remain true for the $y^{(\nu)}$'s. By what we have proved above, the corresponding determinant D_ν is positive. We will be done if we show that $D_\nu \to D$, since by Theorem 7.1 we know $D \neq 0$. This will follow if we show that for each $0 \leqslant j \leqslant m - 1$ and all $1 \leqslant i \leqslant m + K$,

$$L_j B_{i,\nu}(t) \to L_j B_i(t) \tag{7.2}$$

for all t. Now

$$L_j B_{i,\nu}(t) = [y_i^{(\nu)}, ..., y_{i+m}^{(\nu)}] \, L_j g_m(t, \cdot),$$

where $L_j g_m(t, y)$ is 0 if $t < y$, and can be computed from (4.13) for $t \geqslant y$. By the definition of the divided difference (6.2), the divided difference of a function over the points $y_i, ..., y_{i+m}$ is the limit of the divided difference over the points $y_i^{(\nu)}, ..., y_{i+m}^{(\nu)}$ as $\nu \to \infty$. We conclude that (7.2) holds pointwise, and the corollary is proved. (Note: We do not assert that (7.2) holds uniformly, and in general it does not. If, however, at most $m - j - 1$ of the $y_i, ..., y_{i+m}$ are equal to a single value, then this stronger assertion is in fact true.) ∎

We can now show that the matrix M is actually totally positive.

COROLLARY 7.3. *Let $m > 1$. Suppose $t_1 \leqslant \cdots \leqslant t_p$ with at most m of the t's and y's equal to any one value. Then for any $1 \leqslant \nu_1 < \cdots < \nu_p \leqslant m + K$,*

$$D\begin{pmatrix} B_{\nu_1}, ..., B_{\nu_p} \\ t_1, ..., t_p \end{pmatrix} \geqslant 0. \tag{7.3}$$

Strict positivity holds if and only if

$$t_i \in (y_{\nu_i}, y_{\nu_i+m}), \qquad i = 1, 2, ..., p. \tag{7.4}$$

Proof. The fact that D is 0 if (7.4) fails to hold is established easily with Laplace's expansion. We suppose now that (7.4) holds, and show that D is positive by induction on p and on q = number of gaps in the sequence $\nu_1, \ldots, \nu_p$. When $q = 0$, we know that $D > 0$ by Corollary 7.2. Assuming the assertion is true for $p - 1$ and all q and for p and q, we now try to prove it when $\nu_1, \ldots, \nu_p$ has $q + 1$ gaps.

There is no loss of generality in assuming that

$$y_{\nu_i+1} < t_i < y_{\nu_i+m-1}, \qquad i = 1, 2, \ldots, p. \tag{7.5}$$

Indeed, if (7.5) fails, say $t_j \leqslant y_{\nu_j+1}$, then D can be written as the product of two determinants of lower order; viz.,

$$D = D\begin{pmatrix} B_{\nu_1}, \ldots, B_{\nu_j} \\ t_1, \ldots, t_j \end{pmatrix} \cdot D\begin{pmatrix} B_{\nu_{j+1}}, \ldots, B_{\nu_p} \\ t_{j+1}, \ldots, t_p \end{pmatrix}.$$

Suppose now that i denotes one of the missing indices in the sequence $\nu_1, \ldots, \nu_p$ and that l is such that $\nu_1 < \cdots < \nu_l < i < \nu_{l+1} < \cdots < \nu_p$.

To complete the proof, we need a determinantal identity which is useful in the theory of Total Positivity (cf. [8, p. 8]) which in this case reads (cf. [1]):

$$\begin{aligned} &D\begin{pmatrix} B_{\nu_2}, \ldots, B_{\nu_l}, B_i, B_{\nu_{l+1}}, \ldots, B_{\nu_{p-1}} \\ t_1, \ldots, t_{l-1}, t_l, t_{l+1}, \ldots, t_{p-1} \end{pmatrix} D\begin{pmatrix} B_{\nu_1}, \ldots, B_{\nu_p} \\ t_1, \ldots, t_p \end{pmatrix} \\ &\quad = D\begin{pmatrix} B_{\nu_2}, \ldots, B_{\nu_p} \\ t_1, \ldots, t_{p-1} \end{pmatrix} D\begin{pmatrix} B_{\nu_1}, \ldots, B_{\nu_l}, B_i, B_{\nu_{l+1}}, \ldots, B_{\nu_{p-1}} \\ t_1, \ldots, t_l, t_{l+1}, \ldots, t_p \end{pmatrix} \\ &\quad + D\begin{pmatrix} B_{\nu_1}, \ldots, B_{\nu_{p-1}} \\ t_1, \ldots, t_{p-1} \end{pmatrix} D\begin{pmatrix} B_{\nu_2}, \ldots, B_{\nu_l}, B_i, B_{\nu_{l+1}}, \ldots, B_{\nu_p} \\ t_1, \ldots, t_l, t_{l+1}, \ldots, t_p \end{pmatrix}. \end{aligned}$$

Now we may apply the inductive hypothesis to each of the determinants on the right-hand side and to the determinant in front of the desired one on the left-hand side. All of these are positive since in the $p \times p$ determinants the sequences have at most q gaps while by (7.5) the t's lie in the support of the corresponding B-splines. We conclude that $D > 0$. ∎

Theorem 7.1 and Corollary 7.2 were established for the ECT case by entirely different methods (using results on a Green's function; cf. Section 8) by Karlin [8]. The first part of Corollary 7.3 for the ECT case can also be found there. The method of proof of Corollary 7.3 used here comes from deBoor [1].

8. A Green's Function

We saw in Section 4 that the function $g_m(t, y)$ plays a basic role in discussing a one-sided basis for Tchebycheffian splines. In view of (4.3) and (4.4), it is a kind of Green's function, and thus its properties are of interest in their own right. In this section we consider certain determinants formed from g_m, and apply the results to obtain total positivity properties for matrices formed from the one-sided basis for $\mathscr{S}(U_m; \mathscr{M}; \Delta)$.

Suppose that

$$y_1 \leqslant \cdots \leqslant y_p = x_1, \ldots, x_1, \ldots, x_k, \ldots, x_k \tag{8.1}$$

and

$$t_1 \leqslant \cdots \leqslant t_p = \tau_1, \ldots, \tau_1, \ldots, \tau_d, \ldots, \tau_d, \tag{8.2}$$

where each x_i is repeated exactly m_i times and each τ_i is repeated exactly l_i times, with $\sum_1^k m_i = \sum_1^d l_i = p$. Then if $G(t, y)$ is a kernel for which the required derivatives exist, we define

$$G\begin{pmatrix} y_1, \ldots, y_p \\ t_1, \ldots, t_p \end{pmatrix} = \begin{bmatrix} G_{11} & G_{12} & \cdots & G_{1k} \\ G_{21} & G_{22} & \cdots & G_{2k} \\ \vdots & & & \\ G_{d1} & G_{d2} & \cdots & G_{dk} \end{bmatrix}, \tag{8.3}$$

where

$$G_{ij} = \begin{bmatrix} G(t_i, y_j) & L_1{}^*G(t_i, y_j) & \cdots & L_{l_j-1}^*G(t_i, y_j) \\ L_1G(t_i, y_j) & L_1L_1{}^*G(t_i, y_j) & \cdots & L_1L_{l_j-1}^*G(t_i, y_j) \\ \vdots & & & \\ L_{m_i-1}G(t_i, y_j) & \cdots & & L_{m_i-1}L_{l_j-1}^*G(t_i, y_j) \end{bmatrix} \tag{8.4}$$

(the L's operate on G with respect to t and the L^*'s operate on G with respect to the y variable).

The matrix (8.3) can be defined for the kernel $g_m(t, y)$ as long as we require that $1 \leqslant m_i, l_i \leqslant m$, (cf. Lemma 4.1 and (4.14)).

Theorem 8.1. *Let $m > 1$. Suppose that in* (8.1) *and* (8.2) *that at most m t's and y's take on any one value. Then*

$$\det g_m \begin{pmatrix} y_1, \ldots, y_p \\ t_1, \ldots, t_p \end{pmatrix} \geqslant 0. \tag{8.5}$$

Moreover, this determinant is strictly positive if and only if

$$t_{i-m} < y_i < t_i, \qquad i = 1, 2, \ldots, p, \tag{8.6}$$

where the left-hand inequality is ignored if $i \leqslant m$.

Proof. Using Laplace's expansion, it follows that the determinant is zero whenever (8.6) fails (cf. [8, 16, 17]). Suppose now that (8.6) holds, but that the determinant is 0. Then there exist coefficients, not all zero, so that

$$s(t) = \sum_{i=1}^{k} \sum_{j=1}^{m_i} c_{ij} g_{m-j+1}(t, x_i)$$

satisfies

$$L_j s(t_i) = 0, \qquad j = 0, 1,..., l_i - 1 \qquad \text{and} \qquad i = 1, 2,..., k.$$

Suppose l is the maximal index such that s vanishes identically up to y_l. If s does not vanish on any interval to the right of y_l, then it is a nontrivial spline with $p - l + 1$ knots with a total of $m + p - l + 1$ zeros (counting multiplicities); namely, m zeros to the left of y_l, and zeros at the points $t_l ,..., t_p$ (which are not contained in an interval where s vanishes identically by the assumption and the fact that $y_l < t_l$ from (8.6). This is a contradiction of Theorem 5.1.

It remains to consider the case where s vanishes somewhere on an interval to the right of y_l. Suppose r is such that s is identically zero for $y_{r+m} < t$, but does not vanish on any subinterval of (y_l , y_{r+m}). But then, the spline $\tilde{s} = \sum_{i=l}^{r} \sum_{j=1}^{m_i} c_{ij} g_{m-j+1}(t, x_i)$ has an m-tuple zero on the left, an m-tuple zero on the right, and zeros at the points $t_l ,..., t_r$ (which by (8.6) lie in (y_l , y_{r+m})). As $\tilde{s}$ has only $m + r - l + 1$ knots, these $2m + r - l + 1$ zeros again lead to a contradiction of Theorem 5.1. We conclude that the determinant cannot be zero if (8.6) holds.

The fact that the determinant is actually positive under (8.6) follows from a continuity argument exactly as in the proof of Corollary 7.2. ∎

For the ECT case Theorem 8.1 was proved by a complicated multiple induction method by Karlin and Ziegler [17] (see also [8]). We can now give some results on the one-sided basis of section 4. Let

$$\begin{aligned} \tilde{B}_1 ,..., \tilde{B}_{m+K} = u_1 , u_2 ,..., u_m , g_{m-m_1+1}(t, x_1),..., \\ g_m(t, x_1),..., g_{m-m_k+1}(t, x_k),..., g_m(t, x_k). \end{aligned} \tag{8.7}$$

Theorem 4.2 asserts that $\{\tilde{B}_i\}_1^{m+K}$ is a basis for $\mathscr{S}(U_m ; \mathscr{M}; \Delta)$.

THEOREM 8.2. *Let $m > 1$. Suppose $a \leqslant t_1 \leqslant \cdots \leqslant t_{m+K} \leqslant b$ are such that at most m t's and y's take on any one value. Then*

$$D\begin{pmatrix} \tilde{B}_1 ,..., \tilde{B}_{m+K} \\ t_1 ,..., t_{m+K} \end{pmatrix} \geqslant 0. \tag{8.8}$$

Moreover, strict positivity holds if and only if

$$y_i < t_i < y_{i+m}, \qquad i = 1, 2, \ldots, m + K, \tag{8.9}$$

where

$$y_1 \leqslant \cdots \leqslant y_{m+K} = x_0, \ldots, x_0, \ldots, x_k, \ldots, x_k \tag{8.10}$$

with each x_i repeated exactly m_i times (and with $x_0 = a$, $m_0 = m$, and with $y_{m+K+1} \leqslant \cdots \leqslant y_{2m+K}$ arbitrary points larger than b).

Proof. If we apply Theorem 8.1 with $p = m + K$, then by (4.14) we obtain that the determinant formed from the functions

$$u_m, -u_{m-1}, \ldots, (-1)^{m-1} u_1, g_m(\ , x_1), -g_{m-1}(\ , x_1), \ldots, (-1)^{m_i-1} g_{m-m_i+1}(\ , x_1)$$
$$\ldots, g_m(\ , x_k), -g_{m-1}(\ , x_k), \ldots, (-1)^{m_k-1} g_{m-m_k+1}(\ , x_k)$$

is nonnegative and is positive under the conditions (8.9). But if we reorder these functions as in (8.7), the number of interchanges of columns of D exactly accounts for all of the signs, and the result follows. ∎

Now we can prove a total positivity result for the matrix formed from the basis $\{\tilde{B}_i\}_1^{m+K}$.

THEOREM 8.3. *Let $1 \leqslant \nu_1 < \cdots < \nu_p \leqslant m + K$. Then*

$$D\begin{pmatrix} \tilde{B}_{\nu_1}, \ldots, \tilde{B}_{\nu_p} \\ t_1, \ldots, t_p \end{pmatrix} \geqslant 0. \tag{8.11}$$

Strict positivity holds if and only if

$$t_i \in (y_{\nu_i}, y_{\nu_i+m}), \qquad i = 1, 2, \ldots, p. \tag{8.12}$$

Proof. The fact that $D = 0$ when (8.12) fails is established directly using Laplace's expansion. The strict positivity is established exactly as in the proof of Corollary 7.3. ∎

9. AN EXAMPLE

There is no need to consider the well-known cases of trigonometric, exponential, or hyperbolic splines (see Remark 1 in the following section for some references). Instead, in this section we consider an example involving a rather different kind of U_m.

Let $m = 4$. Suppose $p \in C^1[0, 1]$, $p(t) > 0$ on $(0, 1]$, and that $\int_0^1 p^{-1}(t)\, dt < \infty$. Let

$$u_1(t) = 1,$$

$$u_2(t) = \int_0^t p^{-1}(t_2)\, dt_2\,,$$

$$u_3(t) = \int_0^t t_2 p^{-1}(t_2)\, dt_2\,,$$

$$u_4(t) = \int_0^t t_2^2 p^{-1}(t_2)\, dt_2/2.$$

Let $\mathscr{M} = (1, 1, 1,..., 1)$ so that we are considering simple knots, and consider $\mathscr{S}(U_4\,; \mathscr{M}; \Delta)$. This class of splines was used in a scheme for the numerical solution of singular boundary-value problems of the form

$$-(p(t)\, \varphi'(t))'/b(t) + q(t)\, \varphi(t) = f(t)$$

in [24]. This class is a bone fide example of our notion of extended Tchebycheffian spline as here U_m is not spanned by an ECT system (p is not sufficiently smooth, not being in $C^2[0, 1]$, and its behavior is bad at the singular point 0).

To show how the Green's function looks for a specific example, we take the case $p(t) = t^{-\sigma}$, $0 \leqslant \sigma < 1$, as considered in [24]. In this case $u_1(t) = 1$, $u_2(t) = t^{1-\sigma}/(1 - \sigma)$, $u_3(t) = t^{2-\sigma}/(2 - \sigma)$ and $u_4(t) = t^{3-\sigma}/2(3 - \sigma)$. It is also easily checked that $u_1{}^*(y) = 1$, $u_2{}^*(y) = y$, $u_3{}^*(y) = y^2/2$, and $u_4{}^*(y) = y^{3-\sigma}/(3 - \sigma)(2 - \sigma)(1 - \sigma)$. Now, using either (4.2) or (4.13), we may compute

$$g_4(t, y) = \frac{(t^{3-\sigma} - y^{3-\sigma})_+}{2(3 - \sigma)} - \frac{y(t^{2-\sigma} - y^{2-\sigma})_+}{(2 - \sigma)} + \frac{y^2(t^{1-\sigma} - y^{1-\sigma})_+}{2(1 - \sigma)}\,.$$

Local bases for this spline space can now be computed using Theorem 6.2.

10. Remarks

1. Some specific classes of nonpolynomial splines have been considered by various authors. The first seems to be the trigonometric splines considered by Schoenberg [26]. Exponential splines were considered in [31], while hyperbolic splines come up in [30]. These are all examples of Tchebycheffian splines. More recently, Braess, Schaback, Schomberg, and Werner (see [32] for references) have studied various classes of splines which are piecewise

rational functions. These are not Tchebycheffian splines, although some of the basic algebraic facts can be established in this setting.

2. Certain subspaces of the Tchebycheffian splines defined here are of special interest; for example, natural or periodic splines (cf. [13, 17]). Bases and zero properties for these subspaces can easily be developed (cf. [29] for the development in the polynomial case).

3. Monosplines play an important role in development of certain best quadrature formulas, and there are a number of papers on Tchebycheffian monosplines (e.g., see [9, 14, 15, 20, 27]). Clearly, the notion of a Tchebycheffian monospline as studied in these papers (where U_m is spanned by an ECT system) admits of extension to the case considered here.

4. It would be possible to obtain analogs of some of the results presented here in the case where the ties at the knots involve linear combinations of the L_j operators; i.e., a kind of Extended Hermite–Birkhoff type of continuity. For some zero results in the polynomial case, see [29]. A general method for constructing local support bases with such ties can be found in [6].

5. There is not space here for a number of other interesting constructive properties of Tchebycheffian splines. Here we may mention that a generalized Peano representation can be established, and that the analog of Marsden's identity [19], (cf. also [11]) can also be established in this setting. The total positivity properties established here lead, of course, directly to certain variation diminishing properties (cf. [8, 12]).

6. The usual divided difference can be computed recursively by reducing the mth-order one to a difference of $(m - 1)$th-order ones. A similar scheme can be used for generalized divided differences; see [21]. An important computational tool which is missing here is a set of recursions for the stable computation of the B-splines discussed in Section 6 as is available for the polynomial case (cf. [1]).

7. Properties of Green's functions are important in several areas. Some additional references where Green's functions similar to $g_m(t, y)$ are studied include [8, 10, 13, 18], among others.

8. When $\{u_i\}_1^m$ is an ECT-system, the Tchebycheffian splines studied here reduce to the usual ones. We may note that in this case the operators L_j and L_j^* involve ordinary right and left derivatives, respectively. Specifically,

$$D_j = (1/w_{j+1})\, d_R \qquad \text{and} \qquad D_j^* = (1/w_{j+1})\, d_L ,$$

where d_R and d_L are the usual right and left derivatives. We also observe that by a simple argument involving Leibnitz's rule, it is easily seen that the specification of the values $L_0\varphi(t),\ldots, L_j\varphi(t)$ is equivalent to the specification of the values $\varphi(t)$, $d_R\varphi(t),\ldots, d_R{}^j\varphi(t)$ in this case.

9. We have excluded the case $m = 1$ in Theorems 7.1 and 8.1, and their corollaries because their statements are minor variants and because their proofs are so simple. For example, Theorem 7.1 for $m = 1$ is exactly as stated except that in (7.1 we permit equality on the left (all functions considered here are right continuous).

10. The operators $D_j{}^*$ introduced in (4.11) have been taken as limits from the left in order to ensure that (4.14) holds. In particular, in the polynomial case we may observe that $(t - y)_+$ may be differentiated with respect to y at all points t except $t = y$. At this point we may compute either left or right derivatives. But if we want to get $-(t - y)_+^0$, we have to use the left derivative (remember $(t - y)_+^0$ is right continuous).

11. The fact that we "differentiate" $g_m(t, y)$ from the right with respect to t and from the left with respect to y permits us to define mixed derivatives $L_jL_i{}^*g_m(t, y)$ for all $0 \leqslant i, j \leqslant m - 1$. We discuss determinants without the assumption that at most m t's and y's take on any one value in a separate paper.

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